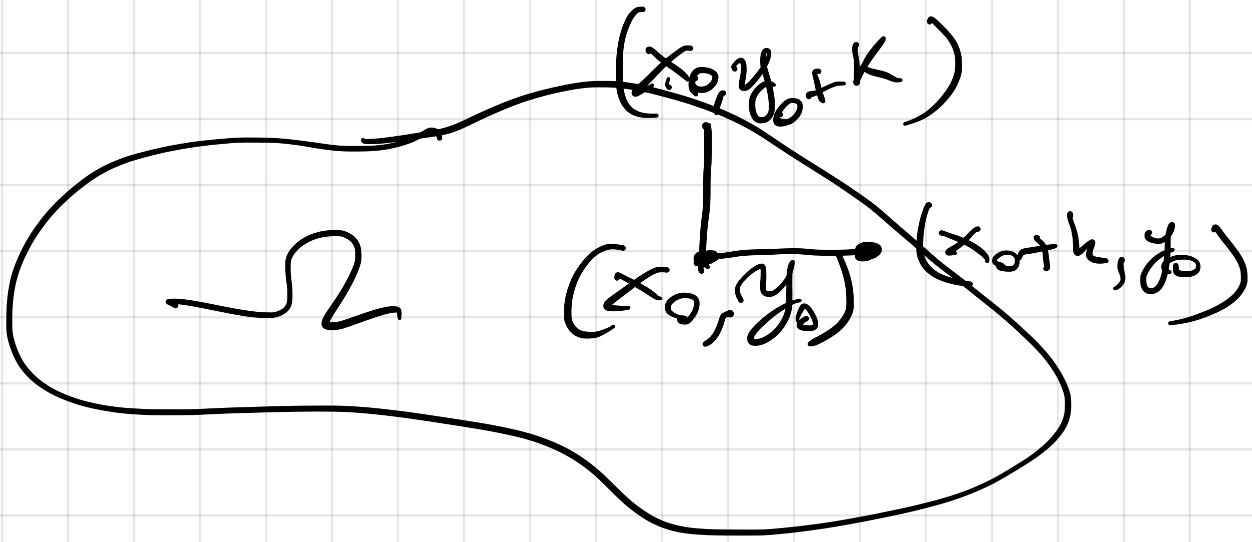
$$f(x,y) \rightsquigarrow \nabla f = (\partial_x f, \partial_y f)$$

$$\partial_x f(x_0,y_0) = f\,\frac{(x_0+h,y_0) - f(x_0,y_0)}{h} \xrightarrow{h \to 0} ?$$



A volte è utile derivare usando le regole note di derivazione in una variabile tenendo l'altra congelata.

<u>Esempio</u> $\partial_x f$ dove

$$f(x,y) = \operatorname{sen}(x^2+y^2)$$

$$\partial_x f = 2x\cos(x^2+y^2)$$

⚠ Se ci sono funzioni non sempre derivabili.

Esempio

$$f(x,y) = |1+x+y|$$

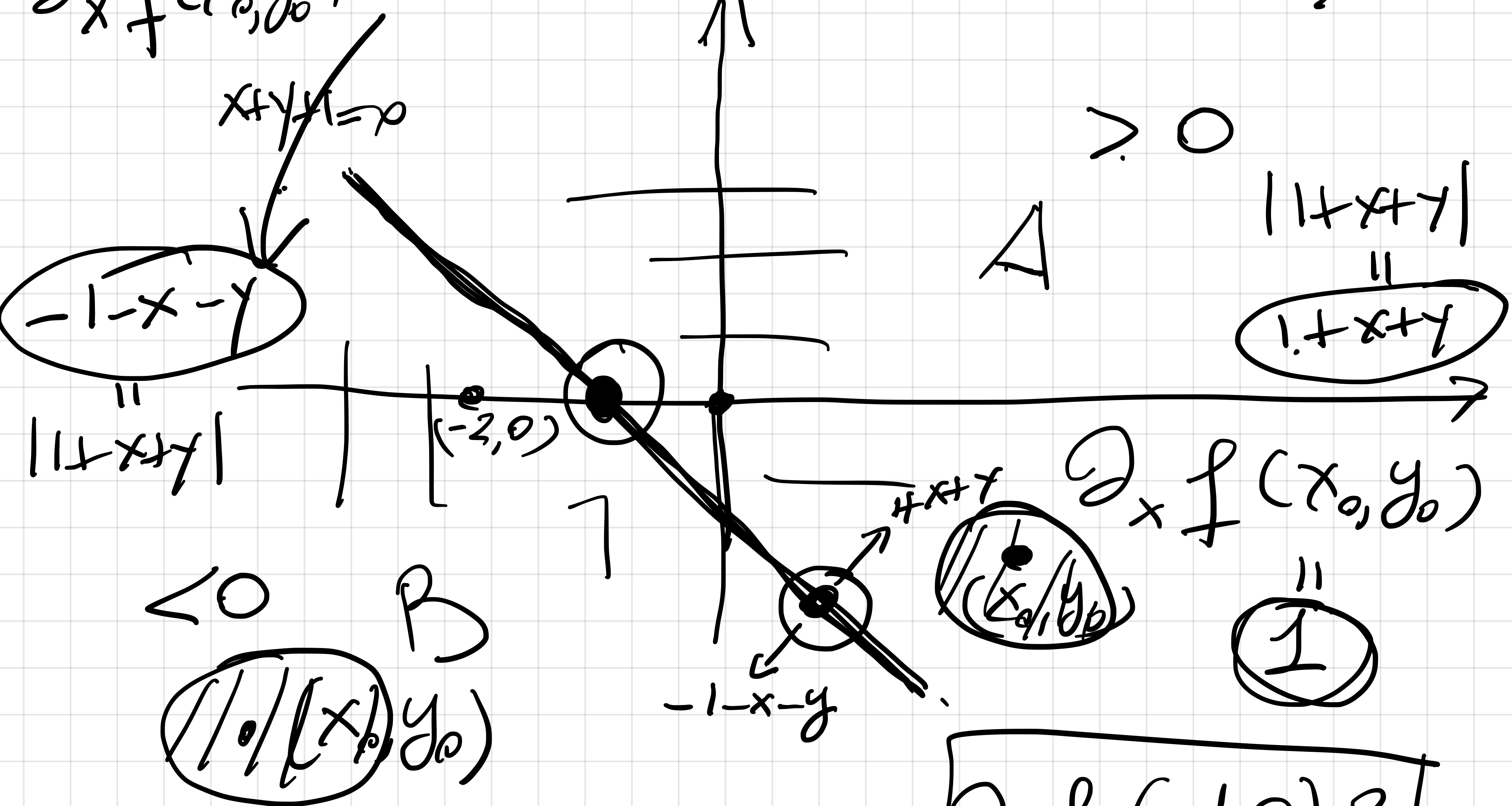
$\partial_x f$? Non fare

$1 \cdot |1+x+y|'$

$1 . 1'$ $\nexists$ in zero!

$\partial_x f(x_0,y_0)$

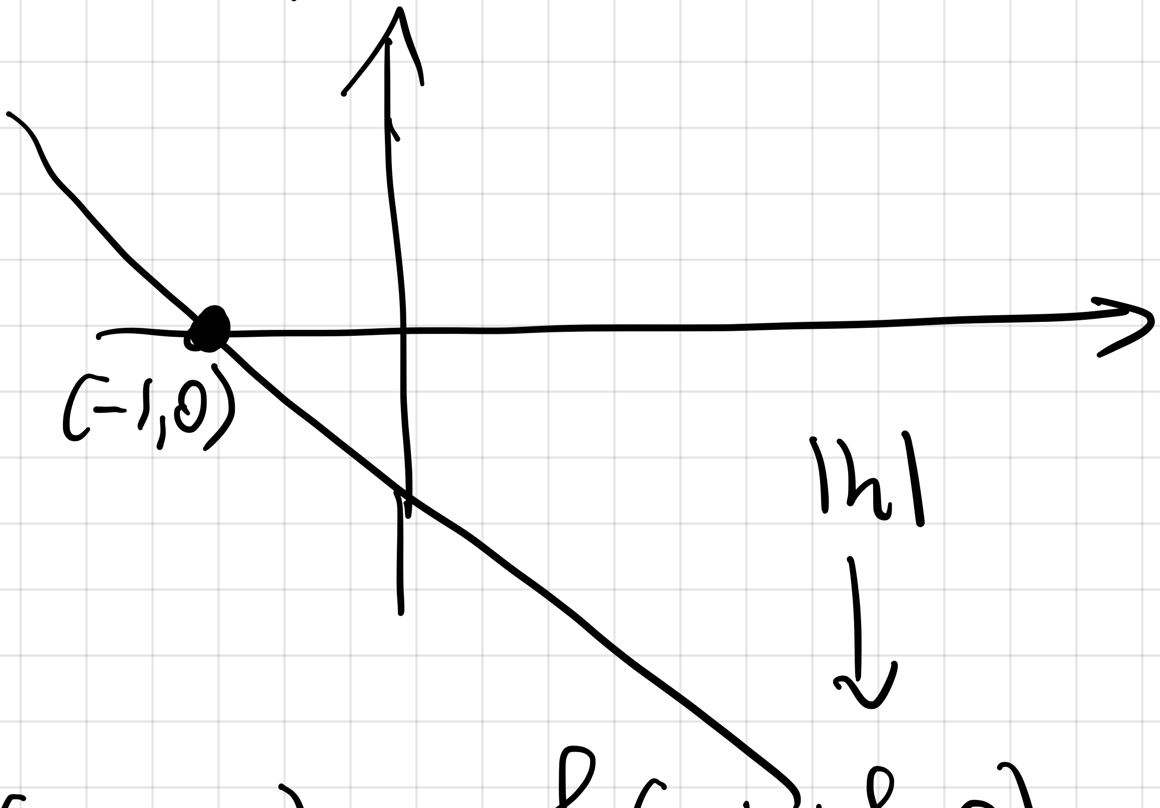
$1+x+y=0$?



$$\partial_x(-1-x-y) = -1$$

$$\boxed{\partial_x f(-1,0)\,?}$$

Il calcolo precedente non funziona ad esempio in $(x_0, y_0) = (-1, 0)$



$$\partial_x f(-1,0) = \frac{f(-1+h,0) - f(-1,0)}{h} \xrightarrow{h \to 0} ?$$

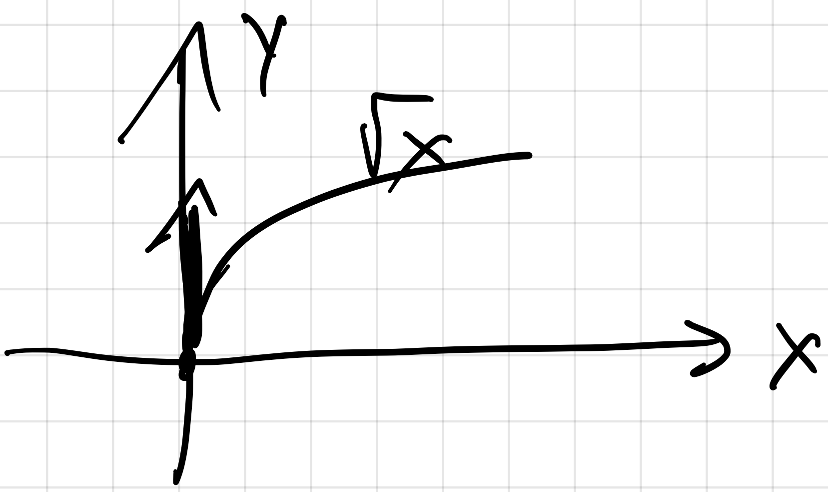
$|h|$ $\quad$ 0

$$f(x,y) = |1+x+y| \Rightarrow f(-1+h,0) = |\cancel{1}\cancel{-1}+h+0| = |h|$$

$$f(-1,0) = |1+(-1)+0| = 0$$

$$\boxed{\partial_x f(-1,0)} = \lim_{h \to 0} \frac{|h|}{h} \nexists$$

Fare attenzione a $\sqrt{}$



$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$\downarrow$

O.K. se $x>0$

(NO) O.K. se $x=0$

<u>Esempio</u> $f(x,y) = \sqrt{1+x^2+y^2}$

$$\partial_x f(x,y) = \frac{2x}{2\sqrt{1+x^2+y^2}}$$ <u>O.K.</u>

$$\partial_x f(1,1) = \frac{2}{2(\sqrt{1+1+1})} = \frac{1}{\sqrt{3}}$$ <u>O.K.</u>

<u>Esempio</u> $f(x,y) = \sqrt{x^2+y^2}$

$$\partial_x f = \frac{2x}{2\sqrt{x^2+y^2}}$$ se $(x,y) = \boxed{(0,0)}$

<u>ATTENZIONE</u>

Se $\partial_x f(1,0) \Longrightarrow \partial_x f(1,0) = \dfrac{2}{2\sqrt{1}} = \boxed{1}$

$$f(x,y) = \sqrt{x^2+y^2}$$

$$\nexists\ \partial_x f(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} =$$

$$= \lim_{h\to 0} \frac{\sqrt{h^2} - \overset{\overset{0}{\shortparallel}}{f(0,0)}}{h} = \lim_{h\to 0} \frac{|h|}{h}\ \nexists$$

DOMANDA: cosa implica $\exists\, \nabla f(x_0,y_0)$?

Teo. (Analisi 1) $f : (a,b) \longrightarrow \mathbb{R}$, $x_0 \in (a,b)$

$$\exists\, f'(x_0) \;\underset{\not\Leftarrow}{\Longrightarrow}\; f \text{ è continua in } x_0.$$

Teo. (Analisi 2)? $f : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x_0,y_0) \in \Omega$

$$\exists\, \big(\partial_x f(x_0,y_0),\ \partial_y f(x_0,y_0)\big) = \nabla f(x_0,y_0)$$

$$\underset{???}{\overset{?}{\Longrightarrow}}\; f \text{ è continua in } (x_0,y_0)$$

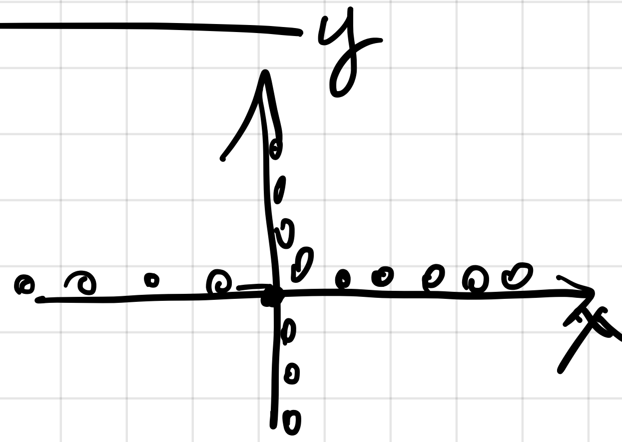
NO!

Esempio $\exists f(x,y)$, $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ t.c.

i) $\exists \nabla f(0,0)$

ii) f è discontinua in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$



Proviamo che f è discontinua in $(0,0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) \begin{array}{l} \nearrow \nexists \\ \searrow \neq f(0,0) \end{array}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \nexists$$

i) $\exists \nabla f(0,0)$

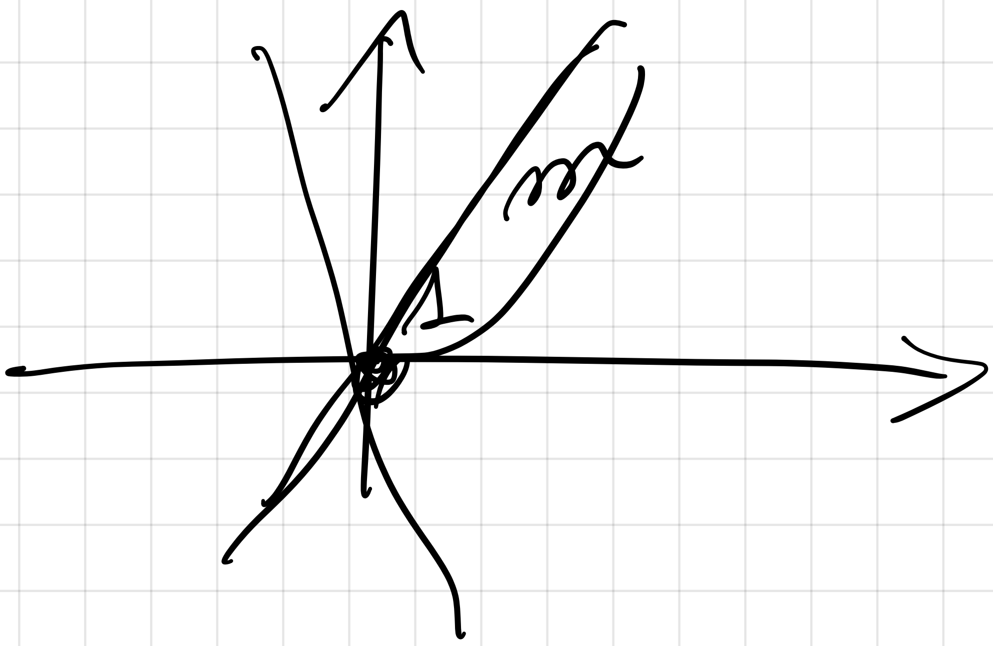
$$\partial_x f(0,0) = \lim_{h\to 0} \frac{\overset{0}{\overset{\shortparallel}{f(h,0)}} - \overset{0}{\overset{\shortparallel}{f(0,0)}}}{h} = \lim_{h\to 0} \frac{0}{h} = \boxed{0}$$

$$\partial_y f(0,0) = \lim_{k\to 0} \frac{\overset{0}{\overset{\shortparallel}{f(0,k)}} - \overset{0}{\overset{\shortparallel}{f(0,0)}}}{k} = \lim_{k\to 0} \frac{0}{k} = \boxed{0}$$

DOMANDA: Trovare la giusta nozione di derivata in 2 o più variabili in modo che queste nozioni implichi la continuità.

~~RICEVIMENTO~~

$$\lim_{(x,y)\to(0,0)} \frac{\ln(|xy|)}{\ln(|x|^2+|y|^2)}$$



$$y = mx$$

$$\frac{\ln(mx^2)}{\ln(2x^2)} =$$

$$= \frac{\ln m + \ln x^2}{\ln 2 + \ln x^2} \xrightarrow{x\to 0\,?}$$

$$\frac{a + \ln x^2}{b + \ln x^2} \qquad \frac{a+t}{b+t} \xrightarrow{t\to -\infty} 1$$

Proviamo altre restrizioni su curve <u>più</u> <u>complicate</u>!

$y = x^2$, $y = x^3$ ...

<u>Polari</u>:

$f(\rho\cos\theta, \rho\sin\theta) =$

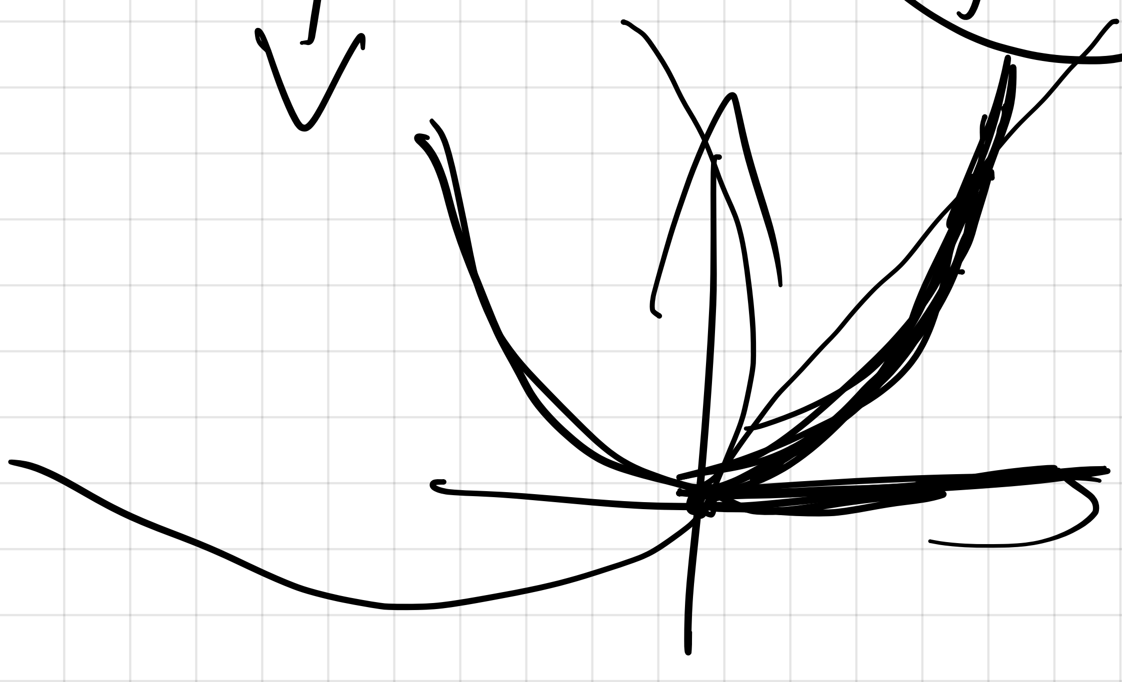
$$= \frac{\ln(|\rho\cos\theta\,\rho\sin\theta|)}{\ln\rho^2} =$$

$$= \frac{\ln(\rho^2 \cancel{+} |\cos\theta \cdot \sin\theta|)}{\ln\rho^2}$$

$$\frac{\ln(\rho^2 \cdot |\cos\alpha||\sin\alpha|)}{\ln \rho^2} =$$

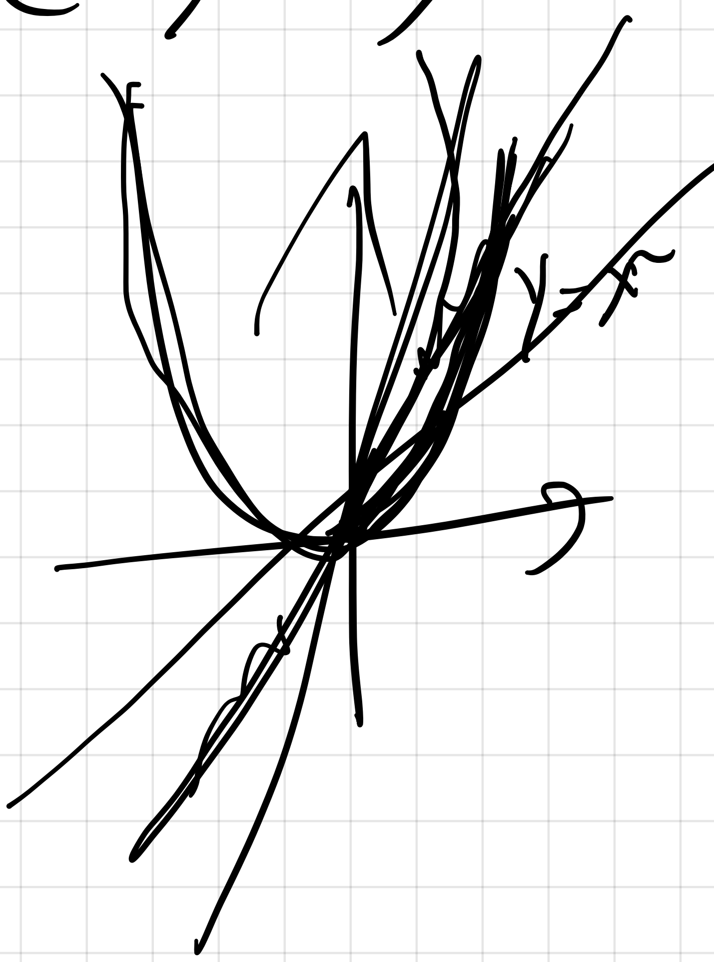
$$\boxed{\frac{\ln(\rho^2) + \ln(|\cos\alpha| \cdot |\sin\alpha|)}{\ln(\rho^2)}}$$

$\downarrow 1$ se α fissato



Proviamo restrizione su $y = x^2$.

$$f(x,y) \rightsquigarrow f(x,x^2)$$



$$\frac{\ln|xy|}{\ln(x^2+y^2)} = \frac{\ln|x|^3}{\ln(x^2+x^4)} = \frac{3\ln(|x|)}{\ln[x^2(1+x^2)]} =$$

$$\frac{3\ln|x|}{\ln\left[x^2(1+x^2)\right]} =$$

$$= \frac{3\ln|x|}{\ln x^2 + \ln(1+x^2)} =$$

$$= \frac{3\ln|x|}{2\ln|x| + \underbrace{\ln(1+x^2)}_{\downarrow 0}} \xrightarrow{x\to 0} \; ?$$

$$\frac{3}{2}$$

$$\lim_{x \to 0} \frac{3 \ln|x|}{2 \ln|x| + \ln(1+x^2)} =$$

$$= \lim_{x \to 0} \frac{3 \cancel{\ln|x|}}{2 \cancel{\ln|x|} \cdot \left(1 + \frac{\ln(1+x^2)}{2 \ln|x|}\right)}$$

$$= \lim_{x \to 0} \left(\frac{3}{2\left(1 + \frac{\ln(1+x^2)}{2 \ln|x|}\right)} \right)$$

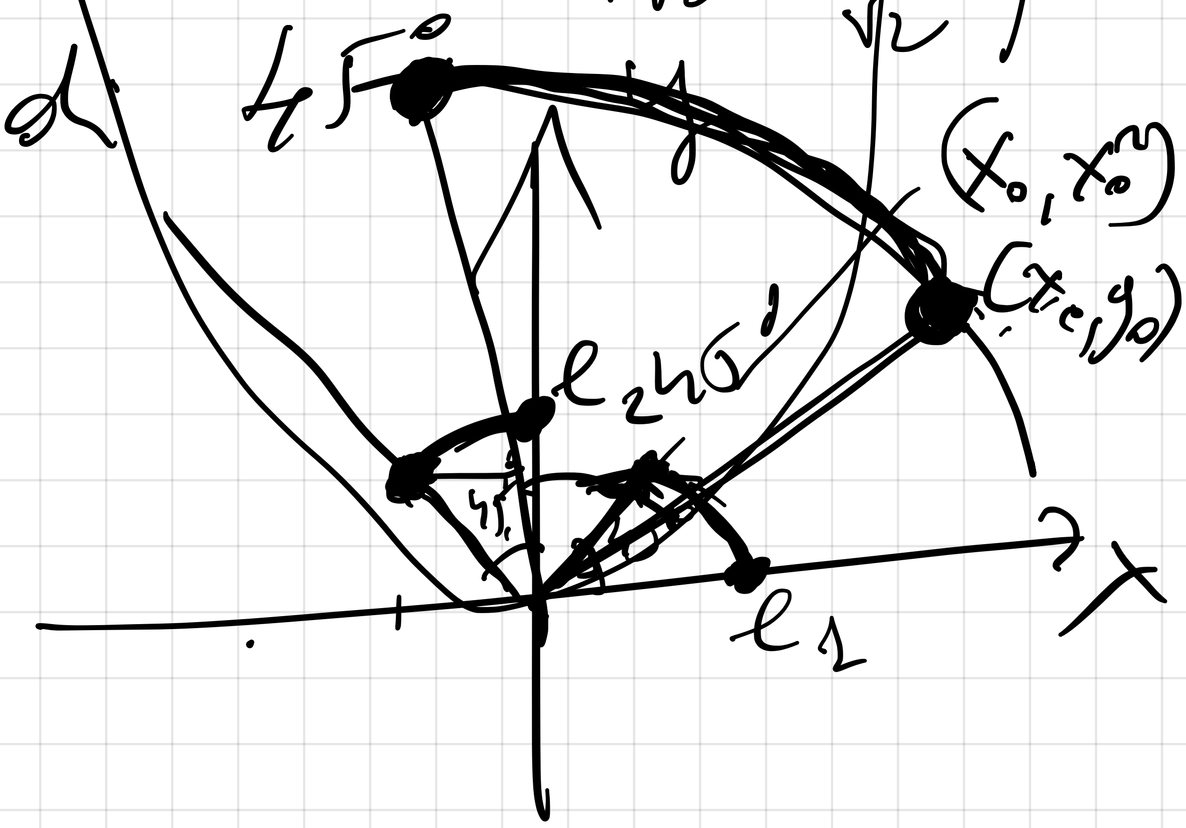
$$\frac{3}{2} \lim_{x \to 0} \frac{1}{\underbrace{1 + \underbrace{\frac{\ln(1+x^2)}{2 \ln|x|}}_{\to 0}}_{\to 1}} \quad \to 1$$

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$L : (e_1, e_2) \longrightarrow (Le_1, Le_2)$$

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \qquad \begin{pmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}$$

L rotazione di 45°



La matrice della rot. di 45° è

$$\underbrace{\begin{pmatrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}}\cdot\begin{pmatrix}x_0\\y_0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt2}x_0 - \frac{1}{\sqrt2}y_0\\ \frac{1}{\sqrt2}x_0 + \frac{1}{\sqrt2}y_0\end{pmatrix}$$

DIFFERENZIABILITÀ

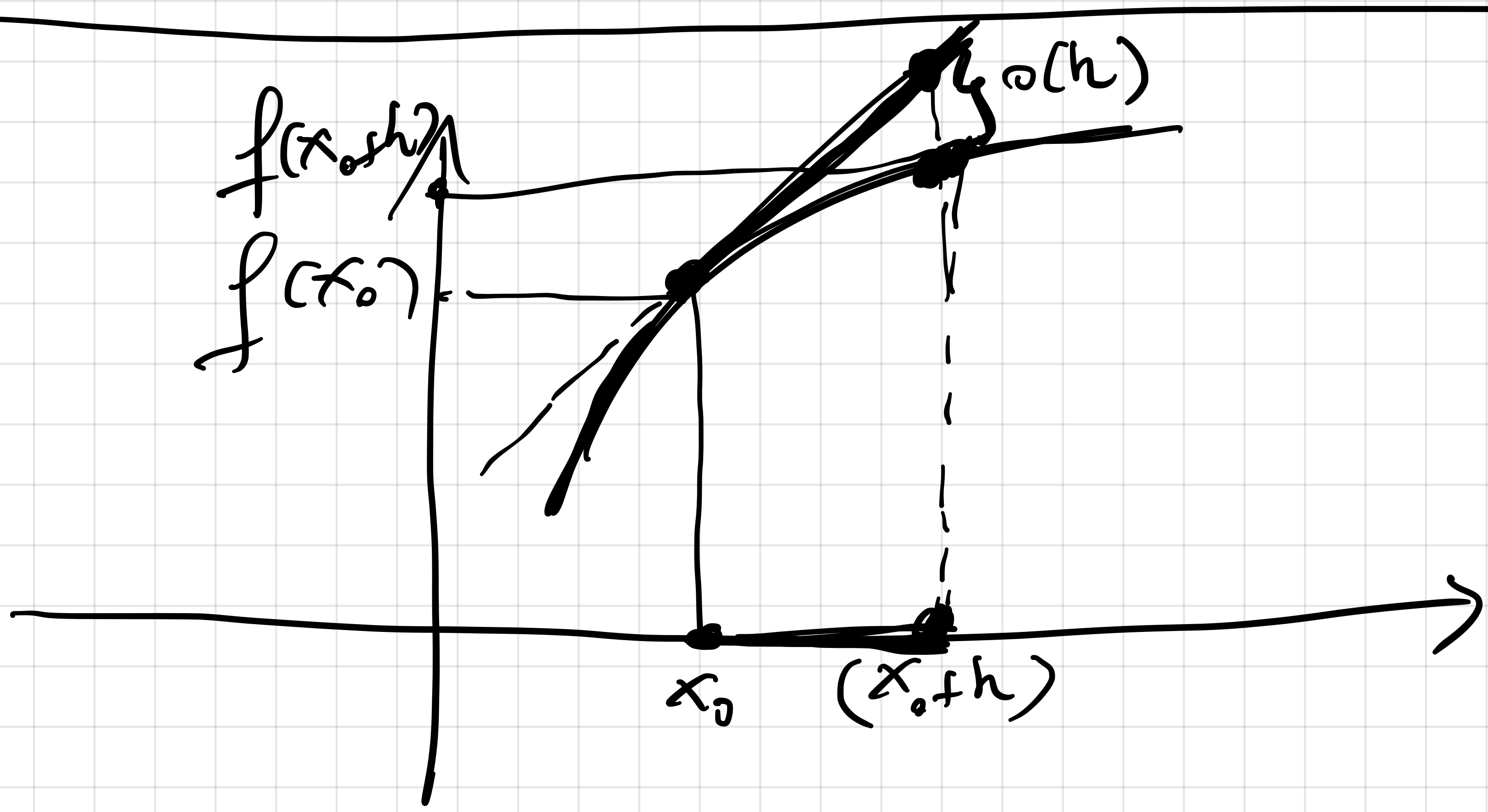
INTERPRETAZIONE GEOMETRICA DELLA DERIVATA IN ANALISI 1

DERIVATA $\Longleftrightarrow \exists$ RETTA TANG. AL GRAFICO

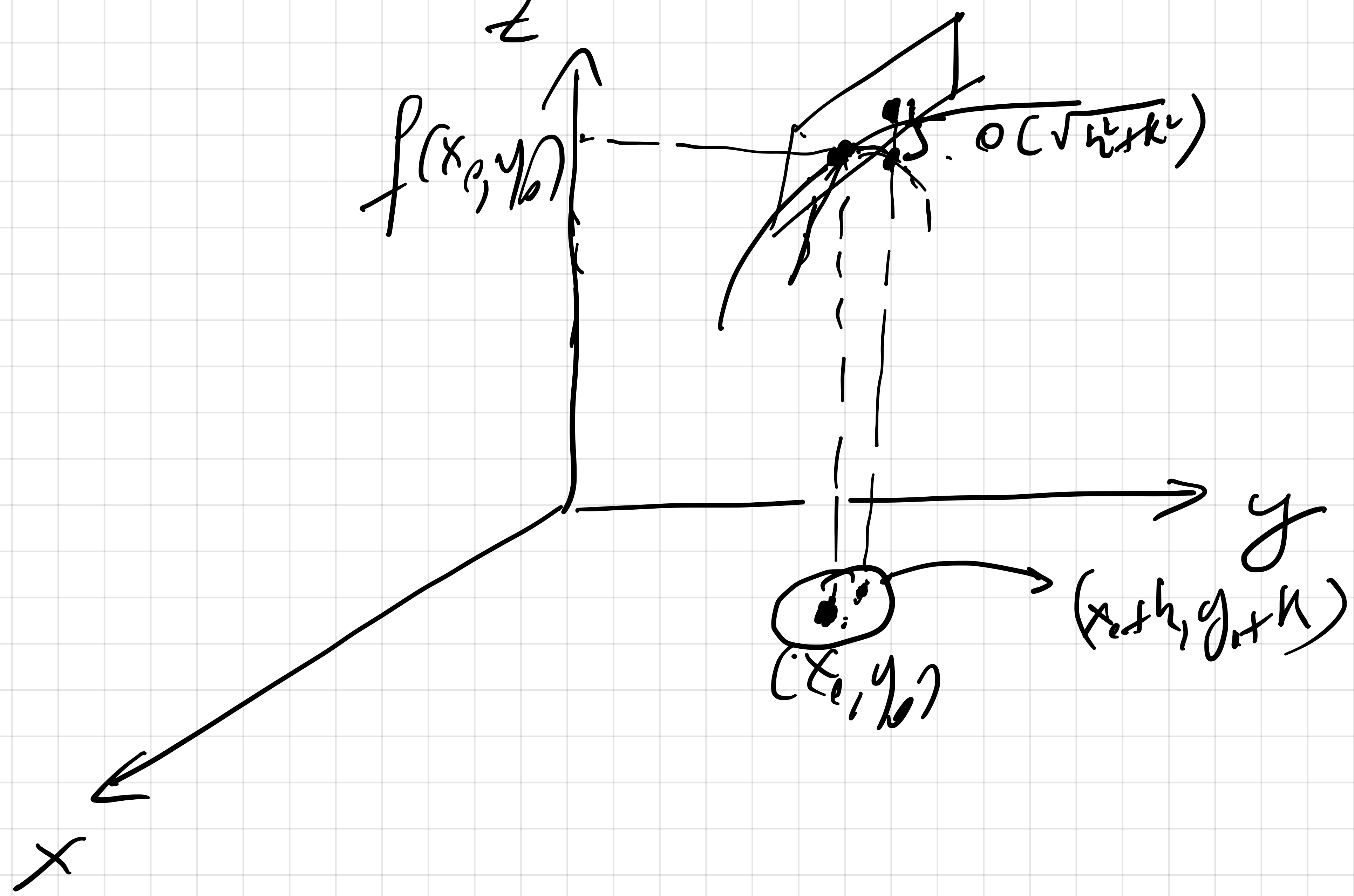
$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$

$$\Updownarrow$$

$$\boxed{\begin{gathered} \underbrace{f(x_0+h)} = \underbrace{f(x_0) + f'(x_0)h} + o(h) \\ \frac{o(h)}{h} \xrightarrow{h \to 0} 0 \end{gathered}}$$



IN PIÙ VARIABILI VOGLIAMO INTRODURRE LA NOZIONE DI PIANO TANGENTE AL GRAFICO



Def. $f : \Omega \subseteq \mathbb{R}^2$ ed $(x_0,y_0) \in \Omega$.

Diciamo che f è differenziabile in (x_0,y_0) se $\exists\, \alpha, \beta \in \mathbb{R}$ t.c.

$$\boxed{f(x_0+h, y_0+k) = f(x_0,y_0) + \alpha h + \beta k + o(\sqrt{h^2+k^2})}$$

Ossia $\exists\, \alpha, \beta \in \mathbb{R}$ t.c.

$$f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha h - \beta k = o\left(\sqrt{h^2+k^2}\right)$$

$$\Updownarrow$$

$$\boxed{\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0}$$

Teo. Se $f: \Omega \to \mathbb{R}$ ed $(x_0, y_0) \in \Omega$, f è diff. in (x_0, y_0)

$\Longrightarrow$ f è cont. in (x_0, y_0).

Teo. Sia $f: \Omega \to \mathbb{R}$ diff. in (x_0, y_0) $\Longrightarrow$ i) $\exists\ \partial_x f(x_0, y_0), \partial_y f(x_0, y_0)$

$\nabla f(x_0, y_0)$

ii) nella def. di diff. $\alpha = \partial_x f(x_0, y_0)$ e $\beta = \partial_y f(x_0, y_0)$

DIM. f diff. in $(x_0,y_0) \Longrightarrow f$ cont. in (x_0,y_0)

(2) $\displaystyle \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

(1) $f(x_0+h, y_0+k) = f(x_0,y_0) + \alpha h + \beta k + o(\sqrt{h^2+k^2})$

$(h,k)\to(0,0)$: $\alpha h + \beta k \to 0$, $o(\sqrt{h^2+k^2}) \to 0$

(1) $\Longrightarrow$ (2)

(2) $\Longleftrightarrow$ $\displaystyle \lim_{(h,k)\to(0,0)} f(x_0+h, y_0+k) = f(x_0,y_0)$

(1)

in (1) prendiamo a sx e a dx $\displaystyle \lim_{(h,k)\to(0,0)}$

a ~~dx~~ trovo $\displaystyle \lim_{(h,k)\to(0,0)} f(x_0+h, y_0+k)$

a dx di (1) $\displaystyle \lim_{(h,k)\to(0,0)} \ldots = \boxed{f(x_0,y_0)}$

$\blacksquare$

Esempio $f(x,y) = x^2 y^3$

È differenziabile in $(1,0)$?

- f è continua in $(1,0)$?

  sì $\Rightarrow$ potrebbe essere diff

  no $\Rightarrow$ non è diff.

f è cont $\Rightarrow$ potrebbe essere diff.

- $\exists \nabla f(1,0)$?

  sì $\Rightarrow$ potrebbe essere diff.

  no $\Rightarrow$ non è diff.

$$\partial_x (x^2 y^3) = 2xy^3 \Rightarrow \partial_x f(1,0) = \boxed{0}$$

$$\partial_y (x^2 y^3) = 3y^2 x^2 \Rightarrow \partial_y f(1,0) = \boxed{0}$$

Quindi la differenziabilità in $(1,0)$ di f equivale a verificare se:

Quindi $\nabla f(0,0) = \underbrace{(0,0)}$

e pertanto il limite finale da studiare diventa:

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{f(1+h, k) - f(1,0) - 0\cdot h - 0\cdot k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{(1+h)^2 k^3 - 0}{\sqrt{h^2+k^2}} =$$

$$= \boxed{\lim_{(h,k)\to(0,0)} \frac{(1+h)^2\, k^3}{\sqrt{h^2+k^2}}} = \boxed{0}$$

$$(1+h)^2 \cdot \frac{k^3}{\sqrt{h^2+k^2}} \to 0$$

$\to 0$ — indet.

$\downarrow$ 1 $\qquad$ $\downarrow$ 0

polari

$$\frac{\rho^{\cancel{3}\,2} \operatorname{sen}^3\theta}{\cancel{\rho}} = \rho^2 \operatorname{sen}^3\theta$$

$$|\rho^2 \sin^3\theta| \leq \rho^2 \xrightarrow{\rho \to 0} 0$$

Esercizio

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}$$

- f non è cont. in $(0,0)$
- $\exists \nabla f(0,0) = (0,0)$

DIFF. IN $(0,0)$?

- NON È DIFF. IN $(0,0)$.

Verifichiamo che f non è diff. in $(0,0)$ usando la def. di diff.

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - \overset{0}{\overset{\|}{f(0,0)}} - \overset{\partial_x f(0,0)=0}{\alpha} h - \overset{=0}{\beta} k}{\sqrt{h^2+k^2}}$$

$$\lim_{(h,k)} \frac{\frac{hk}{h^2+k^2}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{hk}{(h^2+k^2)^{3/2}}$$

$$\lim_{(h,k)\to(0,0)} \frac{hk}{(h^2+k^2)^{3/2}} \overset{?}{=} \nexists$$

polare

$$\frac{\rho^2 \cos\theta \, \mathrm{sen}\,\theta}{\rho^3} = \boxed{\frac{\cos\theta \, \mathrm{sen}\,\theta}{\rho}}$$

$$-\frac{1}{2\rho} \to -\infty$$

$$\frac{\theta}{\rho} = \frac{\cos\theta\,\mathrm{sen}\,\theta}{\rho}$$

$$\frac{\theta}{\rho} = \boxed{\cos\theta\,\mathrm{sen}\,\theta} = \frac{1}{2}$$

$$\theta = \frac{\pi}{4}$$

$$\frac{1}{2\rho} \to +\infty$$